

Classification of quantum degenerate regimes in one-dimensional Bose gases

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Received 26 July 2004 / Received in final form 24 November 2004

Published online 25 January 2005 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2005

Abstract. In this paper we address the different regimes of quantum degeneracy in a one-dimensional Bose gas taking into consideration some parameters that are readily accessible in the experiment. We pay particular attention to the tunability of the trap anisotropy and the number of particles in the system.

PACS. 05.30.Jp Boson systems – 32.80.Pj Optical cooling of atoms; trapping – 67. Quantum fluids and solids; liquid and solid helium – 03.75.Nt Other Bose-Einstein condensation phenomena

1 Introduction

The early studies of low-dimensionality issues in Bose gases were constrained due to the difficulty in realising such systems experimentally. However, with the rapid progress in cooling techniques and the possibility of tightly confine the motion of trapped particles in one or two directions to zero point oscillations, we are facing a scenario where these systems can be actually observed. Some experiments have shown realistic possibilities of creating two- (2D) and one-dimensional (1D) trapped gases [1–4] and achieve quantum degeneracy in these systems. In particular, 1D Bose systems can be formed in atomic integrated optics devices [5], atomic waveguides [6] or elongated atomic traps [7, 8].

The interest in low dimensional Bose systems has been increasing because of recent spectacular experimental advances in cold atoms. Particularly fascinating is the fermionisation of the system in the strongly interacting regime, namely the Tonks-Girardeau gas [9, 10]. In a recent experiment, Paredes et al. [11] have reported the observation of a strongly interacting gas of rubidium atoms in a two-dimensional optical lattice. The strong interaction is reached by adding a third lattice potential along the long axis of the system. This extra potential increases the effective mass of the atoms and thus enhances their interactions. This unprecedented achievement has opened up the possibility of studying all the different regimes of quantum degeneracy that have been identified for a trapped Bose gas confined in one dimension [12]. Generally speaking, these regimes are characterised by the small parameter of the system defined as the ratio between the strength of

the interaction (in 1D) $I = ng_{1D}$ and the kinetic energy $K = (\hbar n)^2/m$, namely [12, 13]:

$$\gamma = \frac{mg_{1D}}{\hbar^2 n}, \quad (1)$$

where m is the mass of the particles, n is the number density and g_{1D} is the one-dimensional coupling strength. The system is in the weakly interacting regime for $\gamma \ll 1$, whereas we enter the Tonks-Girardeau regime for $\gamma \gg 1$. We would like to point out that in this paper we are mainly interested in the quantum degenerate regimes of the system. A discussion about general conditions for a gas to be described as one-dimensional can be found in [14–16].

Let us consider a system of N bosonic atoms confined in an elongated trap where the two tightly confined directions are characterised by the frequencies ω_{\perp} and ω_z in the transverse and longitudinal directions, respectively. We define the trap anisotropy as the ratio between these two frequencies $\lambda = \omega_z/\omega_{\perp}$. Therefore, there are two characteristic harmonic oscillator lengths $L_{\perp}^2 = \hbar/m\omega_{\perp}$ and $L_z^2 = \hbar/m\omega_z$. When the trap anisotropy is much smaller than unity, the geometry of the trap is highly elongated. Olshanii has derived expressions for the 1D scattering amplitude and effective 1D interaction potential for atoms confined in traps of this kind [17]. In terms of the radial extension of the trap, the coupling strength g_{1D} can be expressed as

$$g_{1D} = -\frac{2\hbar^2}{ma_{1D}}, \quad (2)$$

with

$$a_{1D} = -\frac{L_{\perp}^2}{a} \left(1 - \frac{aC}{\sqrt{2}L_{\perp}} \right), \quad (3)$$

where, $C = \lim_{s \rightarrow \infty} \left(\int_0^s ds' / \sqrt{s'} - \sum_{s'}^s 1/\sqrt{s'} \right) = 1.4603\dots$ and a is the three-dimensional scattering length.

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The terms that depend on C correspond to transverse renormalisation effects, and are important only in the case where $|a| \gg L_\perp$, otherwise the coupling strength can be expressed as $g_{1D} = 2a\hbar^2/mL_\perp^2$. If we substitute this coupling strength into equation (1) the expression for the small parameter of the system is given by

$$\gamma = \frac{2am\omega_\perp}{\hbar n}. \quad (4)$$

Notice that the small parameter γ is inversely proportional to the square of the radial extension of the wavefunction. From the above equation it is clear that, for a given number density n , we can change between the weakly and strongly interacting regimes by modifying the scattering length (e.g. via Feshbach resonances [18]) and/or the radial frequency of the trap [19]. Alternatively, it is also possible to modify the effective mass of the atoms as demonstrated in a recent realisation of a Tonks-Girardeau gas [11]. In this paper, we provide a further analysis of the quantum degenerate regimes considering some parameters that can be adjusted in the experiment. In particular we concentrate on the tunability of the radial frequency confinement provided by the experimental set-up [19] and the number of particles in the system. We present a comparison of some of the main length scales involved in the problem, namely the mean interparticle distance L_p , the correlation length [20,21] $L_c = \hbar/\sqrt{nmg_{1D}}$, the radial and longitudinal harmonic oscillator lengths. We re-express some well-known results in terms of variables that can be readily manipulated.

We show that in the boundary where the gas stops having a three-dimensional (3D) behaviour and starts becoming one-dimensional, the correlation length is equal to the radial extension of the wavefunction. This turns out to be equivalent to having an interparticle separation of twice the three-dimensional scattering length a . In the same way, when we change from the one-dimensional weakly interacting regime ($\gamma \ll 1$) to the strongly interacting one ($\gamma \gg 1$), the radial and the longitudinal extensions of the wavefunction are equal and the mean interparticle separation is inversely proportional to the 3D scattering length. The paper is organised as follows: in Section 2 we define and describe the parameters that characterise the different regimes of quantum degeneracy in a one-dimensional Bose gas. Section 3 presents a classification of the degeneracy regimes in terms of the main length scales in the system. Finally in Section 4 we discuss some finite temperature effects that modify the system.

2 Regimes of quantum degeneracy in one-dimensional Bose systems

2.1 Weakly interacting regime

The nature of the quantum degenerate regimes in a trapped 1D gas is strongly influenced by the interparticle interaction and by the presence of the trapping potential.

Considering a cylindrical trap, at sufficiently low temperature T , the radial motion of the particles is “frozen” and is governed by the ground-state wavefunction of the radial harmonic oscillator. In a harmonically trapped 1D gas the temperature of quantum degeneracy is $T_d \simeq N\hbar\omega_\perp$. If the radial extension of the wavefunction is much larger than the radius of interatomic potential, the interaction acquires a 3D character. The decrease of temperature continuously transforms a classical 1D gas to the regime of quantum degeneracy. At $T = 0$ this weakly interacting gas turns into a true Bose-Einstein condensate [12], which is defined as a condensate in which both phase and density fluctuations are suppressed. In the Thomas-Fermi approximation the maximum number density is given by

$$n_0^{(TF)} = \frac{1}{g_{1D}} (\mu - V_{trap}(0)), \quad (5)$$

where μ is the chemical potential and $V_{trap}(z) = m\omega_z^2 z^2/2$. From the normalisation condition of the number density, we can find a relationship between the total number of particles in the system and the chemical potential $\mu = \frac{1}{4} (3\sqrt{2mNg_{1D}\omega_z})^{2/3}$. Following Petrov [12], we define a dimensionless parameter

$$\alpha = \frac{mg_{1D}L_z}{\hbar^2} = \frac{2a}{\lambda L_z}, \quad (6)$$

which is closely related to the anisotropy of the trap and imposes some conditions on the number of particles. As such, this parameter is related to a critical number of particles that allows us to classify the different regimes in the system. For instance, Petrov et al. mention that in the case for $\alpha \gg 1$, the number of particles has to be greater than $N_* = \alpha^2$ [12]. In the case in which α is equal to unity, we are exactly in the region in which the system is neither bosonic nor fermionic. We point out that the descriptions we are dealing with in this paper are in terms of either bosonic theory in the weakly interacting regime or fermionic theory in the strongly interacting one.

The chemical potential can be expressed as

$$\mu = \hbar\omega_z \left(\frac{3N\alpha}{4\sqrt{2}} \right)^{2/3}. \quad (7)$$

In order to be in the Thomas-Fermi (TF) regime, the chemical potential has to be greater than the level spacing of the trap. For $\alpha \ll 1$, the weakly interacting regime is met at any number of particles, and will be described by the TF approximation for all

$$N \gg N_{min}^{(TF)} \Big|_{\alpha \ll 1} = \frac{2\sqrt{2}}{3} \frac{\lambda L_z}{a}. \quad (8)$$

In the case when $\alpha \gg 1$, the system will always be in the TF regime. Using the Thomas-Fermi approximation, the number density at the trap centre can be calculated as $n_{max} = \mu/g_{1D}$ and therefore the small parameter is expressed as

$$\gamma = \left(\frac{4\sqrt{2}}{3} \frac{\alpha^2}{N} \right)^{2/3}. \quad (9)$$

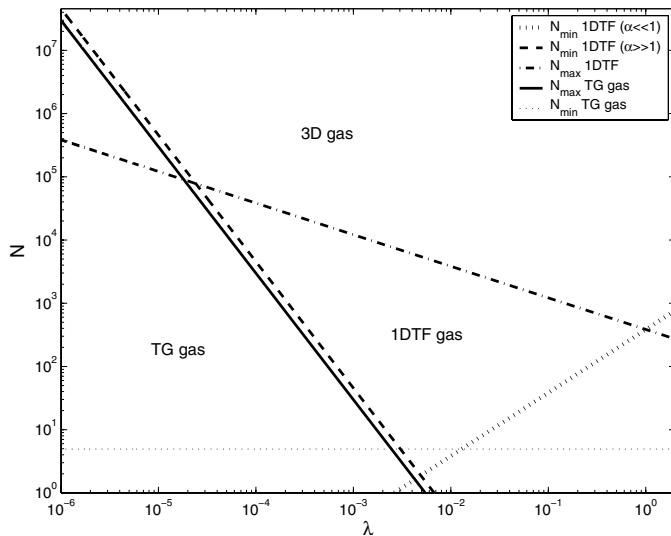


Fig. 1. Number of particles as a function of the trap anisotropy λ , with $\omega_z = 2\pi \times 20 \text{ s}^{-1}$ at $T = 0$ for a system of rubidium atoms. The system has a minimum number of atoms given by the crossing of the two curves for N_{min} . The region enclosed by the curves for N_{min} and N_{max} correspond to a 1D system. The Tonks-Girardeau regime is achieved for tighter confinement. Notice that the more particles we have in a Tonks-Girardeau gas, the tighter the confinement has to be.

From this expression one easily finds that the minimum number of atoms needed to stay in the weakly interacting region of a bosonic mean field is

$$N_{min}^{(TF)} \Big|_{\alpha \gg 1} = \frac{16\sqrt{2}}{3\lambda} \left(\frac{a}{L_{\perp}} \right)^2. \quad (10)$$

Notice how this number of atoms is inversely proportional to the trap anisotropy. In Figure 1, the number of particles as a function of the anisotropy¹ is shown for a system of rubidium atoms with a fixed longitudinal frequency of $\omega_z = 2\pi \times 20 \text{ s}^{-1}$. We include a wide range of values for the trap anisotropy λ for completeness. Notice that to ensure that $\alpha \gg 1$, the value of the anisotropy needs to be smaller than 4.9×10^{-3} , in which case we will have a (weakly interacting) Bose gas in the Thomas-Fermi regime, which in our case means that the radial frequency has to be greater than $2\pi \times 4.06 \times 10^3 \text{ s}^{-1}$. In the case where $\alpha \ll 1$, the minimum number of atoms in the system is given by the crossing of the two curves labelled “ N_{min} 1DTF” which is approximately 2 in this case. However, in a realistic scenario, we need a larger number of particles. We have mentioned above that we are in the TF regime if the chemical potential is greater than the level spacing. In Figure 2 we show the behaviour of the chemical potential normalised to the level spacing of the trap as a function of the trap anisotropy for different number of particles ($N = 6$, $N = 10$ and $N = 3000$). Let us concentrate in the case for $N = 6$; it is clear that when the anisotropy is greater than 1×10^{-2} the TF condition is not met, since the normalised chemical potential goes

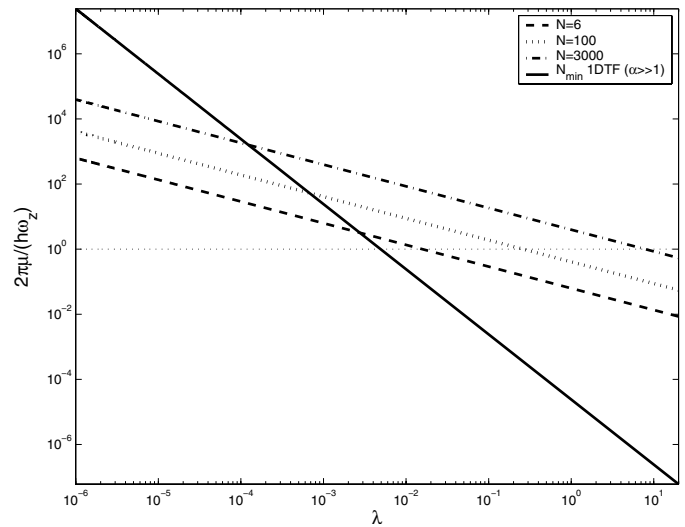


Fig. 2. Chemical potential for systems with different number of particles ($N = 6$, $N = 10$ and $N = 3000$) as a function of the trap anisotropy λ . The curve labelled N for $\alpha \gg 1$ represents the boundary for the one-dimensional Thomas-Fermi gas.

below 1. Notice that the 1D weakly interacting gas with a TF profile in the longitudinal direction exists only in the region above the curve labelled “ N_{min} 1DTF ($\alpha \gg 1$)”. Therefore only this region can be accurately described by the Thomas-Fermi approximation. This can be easily seen if we calculate the value that the small parameter takes below this region. Let us consider the curve for $N = 6$, the system will enter the strongly interacting regime for anisotropies smaller than 2.75×10^{-3} , in such a case, the small parameter given by equation (9) is $\gamma = 1.0077$, and therefore we are on the boundary between the strongly and the weakly interacting regimes. In the case of larger number of particles, for instance $N = 3000$, we are well into the 1D gas description for a wide range of values for the trap anisotropy.

So far we have considered only the axial frequency in the system. However, to be in the one-dimensional Thomas-Fermi (1DTF) regime, the chemical potential will have to be small compared to the radial frequency of the trap. In other words,

$$\frac{\mu}{\hbar\omega_{\perp}} = \lambda \left(\frac{3N\alpha}{4\sqrt{2}} \right)^{2/3} \ll 1. \quad (11)$$

This implies that a system with $N = 1000$ particles will be into the 1D regime for values of the anisotropy smaller than $\lambda = 0.92$; if the longitudinal frequency is fixed to $\omega_z = 2\pi \times 20 \text{ s}^{-1}$, then the radial frequency to fulfil the 1D condition will have to be greater than $\omega_{\perp} = 2\pi \times 21.63 \text{ s}^{-1}$. This imposes a significant constraint on the maximum number of particles for the system to be described by the 1DTF approximation:

$$N \ll N_{max}^{(TF)} = \frac{2L_z}{3a} \sqrt{\frac{2}{\lambda}}, \quad (12)$$

which is also plotted in Figure 1. The area delimited by the curves for N_{min} and N_{max} , correspond to the number

¹ Figures are plotted in *log-log* scale.

of particles needed to have a one-dimensional system as a function of the trap anisotropy λ . This means that the number of particles in the 1D system is bounded as $N_{min} < N < N_{max}$, where N_{min} depends on the value of the dimensionless parameter α . For a trap with an anisotropy $\lambda = 4 \times 10^{-3}$ ($\alpha > 1$), we will need between 3 and 6000 particles approximately to be in the one-dimensional weakly interacting regime.

2.2 Strongly interacting regime

As we have discussed, the strongly interacting regime is achieved when the small parameter of the system is greater than one. In the previous section we have seen that in the case in which $\alpha \ll 1$, the weakly interacting condition is satisfied at any number of particles so long it is greater than $N_{min}^{(TF)} \Big|_{\alpha \ll 1}$. Therefore, we have to concentrate in the case where $\alpha \gg 1$.

It is well-known that in the strongly interacting regime, the gas is actually a Tonks-Girardeau gas whose density profile can be calculated using the Bose-Fermi mapping [10]. The density for a gas under cylindrical confinement in the Tonks-Girardeau regime corresponds to the profile of a gas of non-interacting fermions [22], the maximum density at the centre of the cloud is then given by

$$n_0^{(TG)} = \frac{\sqrt{2N}}{\pi L_z}. \quad (13)$$

The small parameter can then be written in terms of the dimensionless parameter α as

$$\gamma^{(TG)} = \frac{\alpha\pi}{\sqrt{2N}} = \frac{2a\pi}{\sqrt{2N}\lambda L_z}. \quad (14)$$

This implies that the maximum number of particles needed to stay in the Tonks-Girardeau (TG) regime must be

$$N_{max}^{(TG)} = \frac{2}{\lambda} \left(\frac{\pi a}{L_\perp} \right)^2. \quad (15)$$

We point out the clear relationship between the maximum number of particles in the Tonks-Girardeau limit given by equation (15) and the minimum N in the Thomas-Fermi limit given by equation (10). The discrepancy in the prefactor between both expressions comes from the fact that the calculations are considering the system to be well inside the Tonks-Girardeau gas (fermionic theory) or the Thomas-Fermi description (bosonic theory), respectively. In Figure 1 we have plotted the number of particles as a function of the trap anisotropy, notice the gap between the 1D and TG regimes. When the system is in the Tonks-Girardeau regime, the number of atoms must be high enough to ensure its peculiar behaviour. This requirement leads to the trivial condition that we need more than one atom in the system [23]. In our case, equation (15) is only meaningful for $\alpha \gg 1$. This means that, within this theoretical frame we can ensure a noticeable non-ideal behaviour of the TG gas, when the number of particles is greater than $N_{min}^{(TG)} = \pi^2/2 \simeq 5$.

From Figure 1, it is clear that the Tonks-Girardeau regime is achieved for smaller values of the anisotropy (very tight confinement) and lower number of particles than for the 1D or 3D cases. It is clear that the confinement has to be tighter if we want a larger number of particles in the Tonks-Girardeau gas. It is worth mentioning at this point the remarkable experiment it demonstrates the possibility of entering the strongly interacting regime in a 1D degenerate gas with values of γ ranging from 5 to 200 [11].

3 Classification in terms of length scales

In the previous section, we have described some important quantities that determine the behaviour of a one-dimensional Bose gas. As we have seen, the number of particles plays an important role in the description of the system. However, that is not the only option we have to manipulate it. From the definition of the small parameter, it can be seen that we can change the atoms used in the experiment or their scattering properties (e.g. using Feshbach resonances). In such a case, this implies that we are effectively changing the coupling strength g_{1D} . We can also modify the trap anisotropy or modify the effective mass of the atoms as shown in the experiment by Paredes et al. [11]. In that work, the Tonks-Girardeau gas has been obtained in a two-dimensional optical lattice with the addition of an extra periodic potential in the longitudinal direction which effectively increases the value of γ . It is particularly interesting to note that since they are using optical lattices, their small parameter is written in terms of the on-site interaction energy U and the tunnelling amplitude J . In any case, whatever the route we choose to modify the coupling strength, the changes are related to both the radial and longitudinal lengths of the system, and to the correlation length and the interparticle separation. In this section we give an account of the different regimes in the system in terms of length scales as a function of the number of particles and the trap anisotropy.

In the weakly interacting one-dimensional regime at $T = 0$, we have a true Bose-Einstein condensate (BEC) [12]. The Gross-Pitaevskii equation (GPE) provides us with a good description of the condensate wavefunction, which in the Thomas-Fermi approximation gives us the well-known parabolic density profile

$$n^{(TF)}(z) = \frac{\mu}{g_{1D}} \left[1 - \left(\frac{z}{R_{TF}} \right)^2 \right], \quad (16)$$

where μ is given by equation (7) and $R_{TF} = \sqrt{2\mu/m\omega_z^2}$ is the size of the atomic cloud and can be written as

$$R_{TF} = \left(\frac{3L_z^2 N a}{\lambda} \right)^{1/3}. \quad (17)$$

The interparticle separation at the centre of the atomic cloud can be expressed as

$$L_p^{(TF)} = \frac{4}{3} \left(\frac{3L_z^2 a}{N^2 \lambda} \right)^{1/3} \left(1 - \frac{aC}{\sqrt{2}L_\perp} \right)^{-1}. \quad (18)$$

In the case of strong confinement, we can neglect the terms that depend on C , as discussed in the first section. The correlation length for the weakly interacting system is

$$L_c^{(TF)} = \frac{\sqrt{2}}{3} \left(\frac{9L_z^4 \lambda}{Na} \right)^{1/3}. \quad (19)$$

At the crossover between the 3D and the 1DTF gas the number of particles is given by $N_{max}^{(TF)}$. When we substitute this number in the expressions for the main length scales, it is easy to see that the correlation length is equal to the radial size of the wavefunction ($L_c^{(TF)} = L_\perp$). This is quite natural, when the correlation length is greater than the radial size, the macroscopic wavefunction is frozen in the radial direction and we are indeed in the 1D regime. Curiously, this is equivalent to having an interparticle separation given by $L_p^{(TF)} = 2a (1 - aC/\sqrt{2}L_\perp)^{-1}$. This particular value comes from the pseudo-potential approximation used to calculate the atomic scattering in very elongated traps [17]. The mean interparticle separation is related to the 1D density as $L_p = n^{-1}$ [12]. In the case of rubidium in a very elongated trap, $L_p = 2a = 10.4$ nm, giving us a 1D number density of 9.61×10^8 m⁻¹. We would like to emphasize that the intricate interplay between the number of particles, the trap anisotropy and the scattering length, to name a few, allows us to comply with these conditions.

In the three-dimensional description of a dilute Bose gas, condensation occurs when the thermal de Broglie wavelength becomes comparable to the mean interparticle separation [24], which in turn must be larger than the range of the potential (diluteness condition). For binary collisions to happen, the atoms will have to be at a distance within the range of the interatomic potential. These collisions are then better described as waves diffracting off small obstacles and the process is characterised by the scattering length a . When we decrease the number of particles in an elongated trap, the mean interparticle spacing gets larger. In the crossover region, when the number of particles reaches the value $N_{max}^{(TF)}$, the wave scattering description has to account for the one-dimensional nature of the system.

At the boundary with the Tonks-Girardeau gas the number of particles in the system is given by the expression for $N_{min}^{(TF)} \Big|_{\alpha \gg 1}$ and the correlation length and the interparticle separation take the same value $L_c^{(TF)} = L_p^{(TF)} = L_\perp^2/2a$, which we call the *Tonks-Girardeau boundary*. Thus, we have upper and lower bounds for the length scales in the 1DTF description

$$2a \ll L_p^{(TF)} \ll \frac{L_\perp^2}{2a}, \quad (20)$$

$$L_\perp \ll L_c^{(TF)} \ll \frac{L_\perp^2}{2a}. \quad (21)$$

Notice that these conditions depend only on the scattering length and the radial extension of the harmonic oscillator. In what follows, we consider the longitudinal frequency fixed to a value of $2\pi \times 20$ s⁻¹. In Figure 3, we

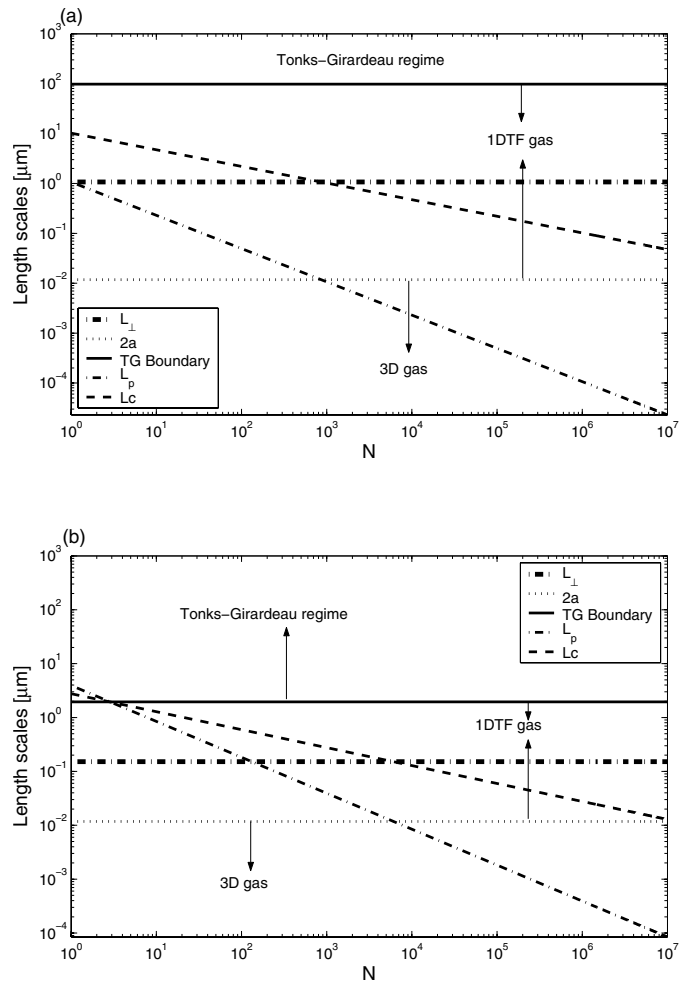


Fig. 3. Main length scales of the system along with the upper and lower bounds for the Thomas-Fermi gas for two different anisotropies (a) $\lambda = 0.2$ and (b) $\lambda = 4 \times 10^{-3}$. We have marked the regimes of the gas as described by the inequalities for the interparticle separation L_p .

have plotted the main length scales of the system along with the upper and lower bounds given above for two different anisotropies (a) $\lambda = 0.2$ and (b) $\lambda = 4 \times 10^{-3}$. For the interparticle separation, the region contained between “ $2a$ ” and “TG Boundary” corresponds to a gas described by the GPE and the density profile is that of a Thomas-Fermi gas. In the case of the correlation length, the corresponding zone is between “ L_\perp ” and “TG Boundary”. Below this region, we are in the three dimensional regime and above it we have a Tonks-Girardeau gas. Notice that the intersections of the interparticle separation and the correlation length with the bounds happens at the same number of particles. In Figure 3a, we can see that according to the behaviour of the interparticle distance L_p , we are in the 3D gas for large number of particles. When this number decreases we enter the 1D regime, since we are in the region mentioned above. However, for smaller values of the trap anisotropy, the 1D region allows us to enter the Tonks-Girardeau regime, as can be seen by the curves for

the interparticle distance and the correlation length going above the Tonks-Girardeau boundary in Figure 3b.

3.1 Strongly interacting regime

The Tonks-Girardeau regime has the counterintuitive characteristic that the system seems to become more interacting when the number density is lower. This can be seen from the definition of the small parameter given by equation (1). From the condition that $\gamma \gg 1$ we obtain that the interparticle distance is inversely proportional to the interaction strength, $L_p^{(TG)} \gg \hbar^2/mg_{1D}$. From equation (13), the interparticle separation in the Tonks-Girardeau gas is $L_p^{(TG)} = \pi L_z/\sqrt{2N}$ and the size of the cloud is $R_{TG} = \sqrt{2N}L_z$. The correlation length is given by

$$L_c^{(TG)} = \sqrt{\frac{\pi\lambda}{a}} \left(\frac{L_z^6}{8N} \right)^{1/4}. \quad (22)$$

Let us consider the system as a chain of N particles arranged in a line of a given length. If the number of particles decreases, the average distance between them is larger and we can achieve the TG gas in the case in which the previous conditions are met. Another possibility to achieve the same result is to change the trap anisotropy, which will allow for more particles to form the TG gas. In this respect, we think that calling this regime “strongly interacting” can be misleading. In principle it is possible to change the interaction strength, for instance, by modifying the scattering length. However, this is not the only possibility we have. We can take a fixed value for the parameter g_{1D} , and we can change either the number of particles, the aspect ratio of the trap or, as it has been demonstrated in [11], the effective mass of the atoms. In the TG regime the system acquires a fermionic character in the sense that whenever two bosons occupy the same position, their wavefunctions vanish, which is only possible in one-dimensional systems. In 2D or 3D systems with N particles, we can hold $N - 1$ of them fixed and move the remaining one throughout the system without encountering any of the fixed particles, but in 1D the motion of this particle is blocked by the rest. According to this description, the TG gas is similar to a gas of non-interacting fermions, for example a gas of free electrons in one-dimension. An electron of mass m_e is confined in a line of length L by infinite barriers and its wavefunction $\psi_n(x)$ is a solution of the Schrödinger equation $\hat{H}\psi = E\psi$. When we accommodate N electrons on this line, we have to take into account the Pauli exclusion principle. In other words, each orbital can be occupied at most by one electron. A pair of orbitals labelled with the quantum number n can accommodate two electrons, one spin up and one with spin down. In the same way, the atoms in a Tonks-Girardeau gas will have to “interact strongly” to mimic the behaviour imposed in the fermionic case by the exclusion principle, avoiding collisions and preventing their wavefunctions from vanishing.

In the regime of strong interactions, the correlation length is much smaller than the mean interparticle separation. Thus, at the crossover from the 1DTF gas to

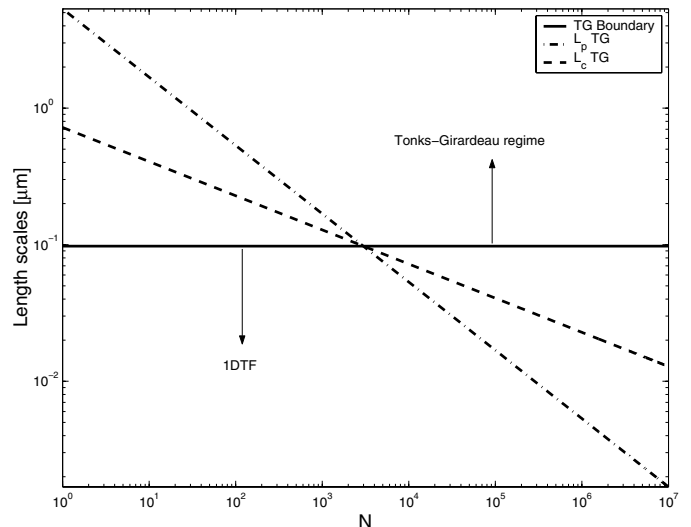


Fig. 4. For a trap anisotropy $\lambda = 2 \times 10^{-4}$, the Tonks-Girardeau gas can be achieved for greater number of particles. Notice that in the regime of strong interactions the correlation length is smaller than the interparticle distance.

the strongly interacting regime ($N = N_{max}^{(TG)}$), the correlation length is equal to the interparticle distance ($L_c^{(TG)} = L_p^{(TG)}$), equivalently

$$\{L_c^{(TG)}, L_p^{(TG)}\} \gg \frac{L_{\perp}^2}{2a}. \quad (23)$$

Figure 4 shows the length scales of the system as a function of the number of particles for a smaller value for the trap anisotropy ($\lambda = 2 \times 10^{-4}$). It is clear that we can attain the strongly interacting regime for a larger number of particles than in the previous cases and therefore there are better chances to observe this phase experimentally. We also point out that the region that describes the 1D weakly interacting gas gets narrower and this situation will be discussed below.

In Figures 5 and 6, we have plotted the main length scales in the system as a function of the trap anisotropy ($\omega_z = 2\pi \times 20 \text{ s}^{-1}$) for (a) $N = 1 \times 10^3$ particles and (b) $N = 1 \times 10^4$ particles, respectively. For the weakly interacting case (in Fig. 5), the region for the 3D gas is marked by the intersection of the correlation length L_c and the radial size L_{\perp} . To the left of this region the system becomes 1D and to the left of the TG boundary we have a Tonks-Girardeau gas. The range of values for the trap anisotropy in which the gas has three-dimensional characteristics is bigger for larger number of particles (see Fig. 5b). This means that the one-dimensional Thomas-Fermi gas will be attained more easily when the system has less particles (see Fig. 5a).

For the sake of clarity, we have plotted the strongly interacting counterpart of the system in Figure 6a for $N = 1 \times 10^3$ particles and Figure 6b for $N = 1 \times 10^4$ particles. As expected, the TG boundary is given by the intersection of the interparticle distance and the correlation length. We point out that the change in the number of particles only

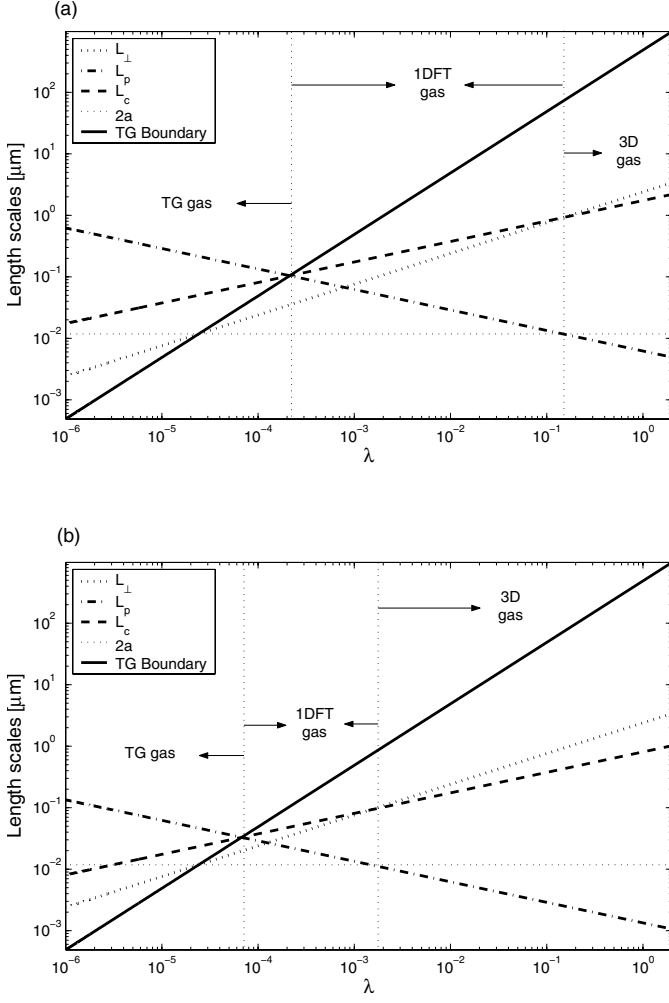


Fig. 5. Main length scales for a weakly interacting system of rubidium atoms in an elongated trap ($\omega_z = 2\pi \times 20 \text{ s}^{-1}$) as a function of the trap anisotropy, for (a) $N = 1 \times 10^3$ particles and (b) $N = 1 \times 10^4$ particles, respectively. Notice that for smaller number of particles in the system, the range of values for the trap anisotropy that satisfy the one-dimensional conditions is larger.

shifts slightly the boundary of the transition. From this discussion, we can understand the crossing of these length scales to indicate a transition from one degenerate regime to another one.

It is particularly interesting to see what happens when we match the boundary conditions described above for the weakly and strongly interacting cases, which implies that the 1D Thomas-Fermi region collapses. Referring to Figures 3 and 5, this corresponds to having $L_p = L_c = L_\perp$, which implies that the radial frequency is given by

$$\omega_\perp^{(3D \rightarrow TG)} = \frac{\hbar}{4a^2m}, \quad (24)$$

this is equivalent to a trap anisotropy given by $\lambda^{(3D \rightarrow TG)} = 4a^2/L_z^2$. Figure 7 shows this situation for the system we have been considering. In this case, we go directly from the three-dimensional gas into the Tonks-Girardeau regime. For a system of rubidium atoms with

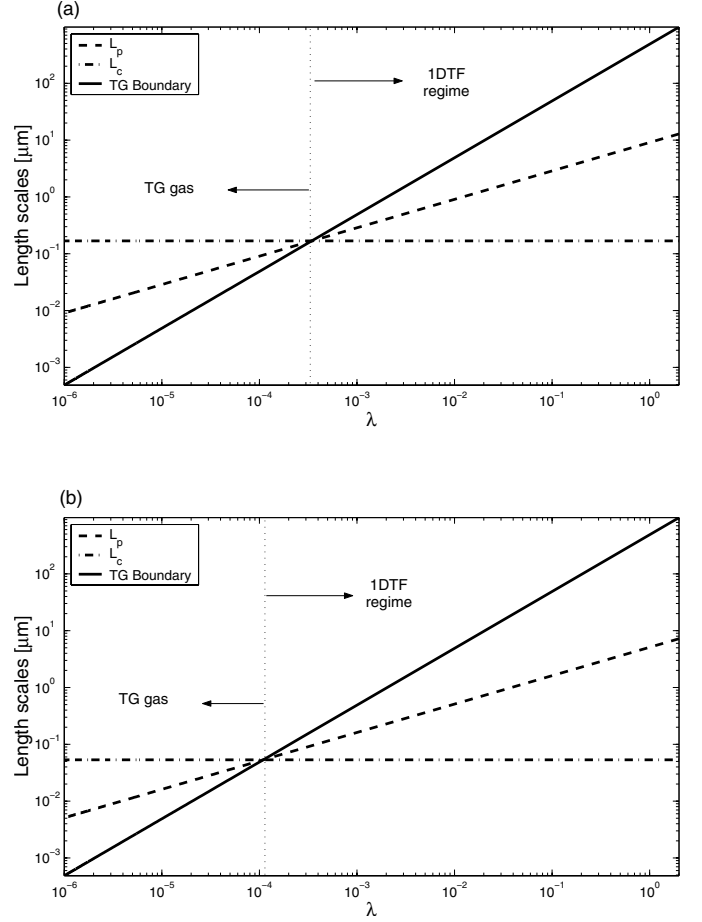


Fig. 6. Main length scales for a strongly interacting system of rubidium atoms in an elongated trap ($\omega_z = 2\pi \times 20 \text{ s}^{-1}$) as a function of the trap anisotropy, for (a) $N = 1 \times 10^3$ particles and (b) $N = 1 \times 10^4$ particles, respectively. The change in the number of particles shifts the boundary between the Tonks-Girardeau gas and the 1D Thomas-Fermi gas only slightly.

a longitudinal frequency $\omega_z = 2\pi \times 20 \text{ s}^{-1}$, we will need a radial frequency of approximately $2\pi \times 830 \times 10^3 \text{ s}^{-1}$ to enter the Tonks-Girardeau regime without having a one-dimensional Thomas-Fermi gas and the number of particles will have to be smaller than $N_{max}^{(TF)} = N_{min}^{(TF)} \simeq 78000$. For greater radial frequencies, equations (20) and (21) are not valid anymore.

4 Finite temperature effects

In the discussion of Sections 2 and 3, we have examined the quantum degenerate regimes of a gas of ultracold bosons overlooking the role of temperature; in other words we have considered the case in which $T = 0$. Nevertheless, temperature is a crucial parameter in the description of the system. For instance at very low temperatures, the phase fluctuations in the system can be suppressed and we will end up with a true BEC [12,25]. Generally speaking, we can identify three different regimes at temperatures $T \ll T_d$, where $T_d = N\hbar\omega_z/k_B$ is the degeneracy

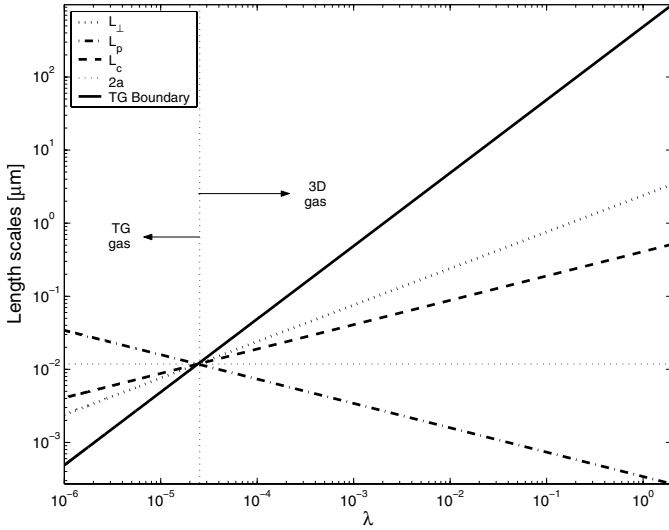


Fig. 7. Main length scales for a system of rubidium atoms in an elongated trap, with the longitudinal frequency $\omega_z = 2\pi \times 20 \text{ s}^{-1}$. When the trap anisotropy reaches a value $\lambda = 2.41 \times 10^{-5}$, we have a transition directly from the 3D gas to the Tonks-Girardeau gas. This corresponds to having $L_p = L_c = L_\perp$.

temperature and $k_B = 1.3807 \times 10^{-23} \text{ J/K}$ is the Boltzmann constant. In the case of the one-dimensional Thomas-Fermi gas, by lowering the temperature we can suppress fluctuations in both phase and density, leading to the existence of a true Bose-Einstein condensate. For a certain temperature, phase fluctuations will still remain in the system giving rise to a quasi-condensate phase [12], the characteristic temperature in this case is given by

$$T_\phi = 2T_d \left(\frac{\lambda L_z}{3Na} \right)^{2/3}. \quad (25)$$

The Tonks-Girardeau gas will be achieved for any temperature much smaller than T_d but only when the number of particles is small enough, and that number depends on the trap anisotropy as shown in Figure 1. The dynamics of the system will be predominantly one-dimensional when the temperature is smaller than the energy of the lowest radial excitation, in other words when

$$k_B T \ll k_B T_\perp = \hbar\omega_\perp. \quad (26)$$

In Figure 8, we have plotted the number of particles as a function of temperature according to the expressions for the degeneracy temperature T_d and the phase temperature T_ϕ for two different trap anisotropies (a) $\lambda = 1 \times 10^{-2}$ and (b) $\lambda = 1 \times 10^{-3}$, respectively. In order to understand the different regimes we have marked the limits given by the $N_{max}^{(TG)}$ for the Tonks-Girardeau gas and $N_{max}^{(TF)}$ for the 1DTF gas obtained in Section 2. The vertical line in both figures represents the temperature T_\perp given by the one-dimensionality condition (26) and are $T_\perp \simeq 95.6 \text{ nK}$ in Figure 8a and $T_\perp \simeq 956 \text{ nK}$ in Figure 8b. In the case of $\lambda = 1 \times 10^{-2}$ the anisotropy is simply not enough to permit the formation of a Tonks-Girardeau gas. It is clear that

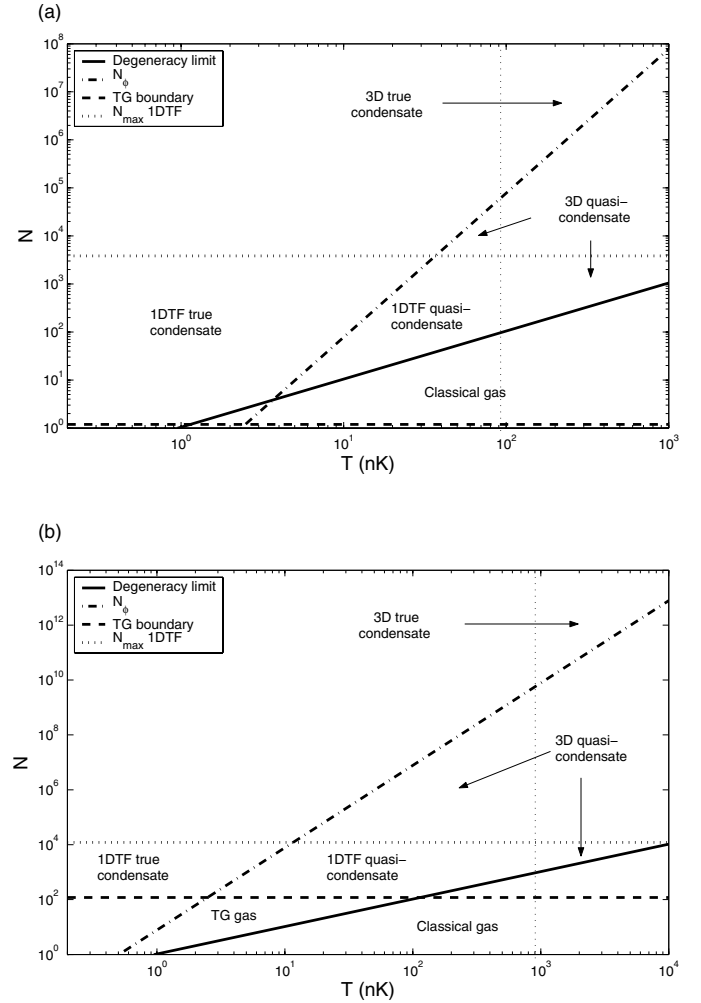


Fig. 8. Number of particles as a function of temperature for trap anisotropies (a) $\lambda = 1 \times 10^{-2}$ and (b) $\lambda = 1 \times 10^{-3}$, respectively. Notice that at smaller trap anisotropies, the Tonks-Girardeau regime is much bigger.

the decrease in temperature results in a continuous change from a classical gas to a quantum degenerate one. For sufficiently low number of particles this transformation takes us from a classical gas through a 1DTF quasicondensate to a 1D true condensate.

When the number of particles is greater than $N_{max}^{(TF)}$, we enter the three-dimensional regime. It has been shown that for very elongated 3D bosonic systems it is possible to have a phase fluctuating three-dimensional condensate [25]. The regions of 3D quasicondensate and true condensate can be easily identified in our phase diagram (Fig. 8).

Figure 8b shows how the system changes when the trap anisotropy has a smaller value, in this case we chose a trap anisotropy $\lambda = 1 \times 10^{-3}$. The most striking difference is the possibility to enter the strongly interacting regime marked in the diagram as the “TG gas” region, in this case at temperatures between 1 nK and 100 nK and $N_{max}^{(TG)} \simeq 120$ particles.

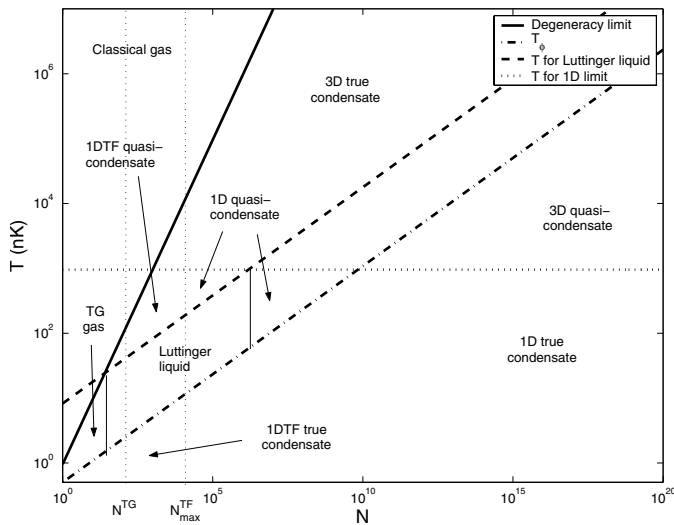


Fig. 9. Phase diagram of a trapped one-dimensional Bose gas for a trap anisotropy $\lambda = 1 \times 10^{-3}$. For temperatures satisfying conditions (26) and (27), the system is in the Luttinger liquid regime.

In Figure 8b there is a boundary between the 1D TF quasicondensate and the Tonks-Girardeau gas in which the system starts acquiring fermion properties. Near this boundary ($N \simeq 10^2$) the behaviour of the gas is neither purely bosonic nor purely fermionic. The connection between fermion and boson statistics in 1D system is a well-known fact, and in this respect, the harmonic-fluid approach has served as a framework to show the similarities of one-dimensional Bose and Fermi fluids. A very good example is the so-called Luttinger liquid, characterised by a linear spectrum of gapless excitations [26]. Monien et al. have shown that under appropriate experimental conditions a trapped one-dimensional Bose gas can be described as a Luttinger liquid [27]. The behaviour of this system is basically a finite temperature effect which is relevant when

$$k_B T \ll k_B T_{LL} = 2\pi\hbar\omega_z \left(\frac{3Na}{2\sqrt{2}L_z\lambda} \right)^{1/3}. \quad (27)$$

For three-dimensional traps ($L_z = L_\perp$) the gap in the lowest mode is discernible and Bose-Einstein condensation is possible. However, when condition (27) is satisfied, the gapless mode forbids the formation of a condensate at finite temperatures. Figure 9 shows the same situation as in Figure 8b ($\lambda = 1 \times 10^{-3}$), except that in this case we have decided to plot the temperature as a function of the number of particles. The two vertical lines correspond to the number of particles that define the boundaries for Tonks-Girardeau gas and 1D Thomas-Fermi gases; in this case $N_{TG} \approx 120$ particles and $N_{max}^{TF} \approx 12000$ particles, respectively. For temperatures satisfying conditions (26) and (27), the system is in the Luttinger liquid regime. Notice that this region spans the area between the strongly and weakly interacting regimes, which is precisely where the system acquires a Fermi character. This phase dia-

gram is in agreement with the results presented by Monien in reference [27] in that at lower temperatures, and with the appropriate number of particles, the Luttinger liquid behaviour is washed out by the finite-size gap in the excitation spectrum and Bose condensation is possible once again.

5 Summary

In this paper, we have addressed the different regimes of quantum degeneracy in a one-dimensional Bose gas considering some of parameters that are readily accessible in the experiment. In particular we concentrated on the number of particles in the system and the tunability of the ratio of the trapping frequencies. The number of particles plays a crucial role in the description of the system and we have given some expressions for the limiting number of particles in the different regimes of quantum degeneracy. The interplay of the number of particles and the tightness of the trapping potential allows us to have certain control over the transitions in the system. A clear example is the fact that at lower number densities is it possible to access the strongly interacting regime, provided that there is a tight enough confinement.

We have also presented a classification of these degenerate regimes in terms of the different length scales involved in the problem, in other words the interparticle distance L_p , the correlation length L_c and the radial and longitudinal harmonic oscillator lengths. We showed that in the boundary where the gas stops having a three-dimensional behaviour and starts becoming 1D, the correlation length is equal to the radial extension of the wavefunction. This turns out to be equivalent to having an interparticle separation of twice the three-dimensional scattering length a . In the same way, when we change from the one-dimensional weakly interacting regime ($\gamma \ll 1$) to the strongly interacting one ($\gamma \gg 1$), the radial and the longitudinal extensions of the wavefunction are equal and the mean interparticle separation is inversely proportional to the 3D scattering length. The boundaries introduced here allow us to identify the experimental conditions to access different regimes, for instance the possibility of attaining the Tonks-Girardeau regime without having a 1D Thomas-Fermi (quasi-)condensate.

Finite temperature in the system introduces some modifications in the description, for example we can talk about the quenching of phase fluctuations at sufficiently low temperatures. We have included the effects of temperature in our phase diagram to have a more complete picture of the transitions that can be observed in the system. In particular, we have discussed the possibility of observing nontrivial behaviour of an interacting quantum gas under suitable experimental conditions.

I would like to thank Prof E.A. Hinds for useful discussions. This work has been supported by the QGATES project.

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